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On Path Connected Graphs

MORDECHAI LEWIN

Department of Mathematics, The Technion, Israel Institute of Technology, Haifa

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Given n and i, $n > 2$, $2 \leqslant i \leqslant n-1$, the smallest size of an n-graph without endvertices is obtained, which ensures a path of length i between any two vertices of the graph.

1. INTRODUCTION

In dealing with hamiltonian graphs Oystein Ore introduced *Hamilton-connectedness* of graphs [9]. A graph is *Hamilton-connected* (or *H-connected*) if there is a hamiltonian path from every vertex to every other vertex in the graph. An account of results on H-connected graphs is to be found in [1]. In [5] Faudree, Rousseau and Schelp generalize the notion of H-connectedness to P_i-connectedness. A graph is P_i*-connected* or simply P_i, if there is a path of length i (number of edges in the path) between any two distinct vertices of the graph, where i is a positive integer less than n, the order of the graph. H-connectedness of a graph G then means that G is P_{n-1}.

In [6] Faudree and Schelp generalize a result of Ore by showing that Ore's sufficient conditions for H-connectedness actually ensure P_i for every i, $4 \leqslant i \leqslant n-1$.

In this paper we wish to generalize a similar result of Ore stating how many edges in a graph ensure H-connectedness. For each n and for each i, $2 \leqslant i \leqslant n-1$, we obtain best possible lower bounds on the number of edges in a graph without endvertices ensuring P_i. (For $i = 1$ the situation becomes trivial).

We shall refer to two results of Ore stated as

THEOREM 01. (Ore [8]). *Let G be a simple graph with n vertices and m edges, such that $m \geqslant \frac{1}{2}(n-1)(n-2) + 2$. Then G is hamiltonian.*

THEOREM 02. (Ore [9]). *Let G be a simple graph with n vertices and m edges, such that $m \geqslant \frac{1}{2}(n-1)(n-2) + 3$. Then G is H-connected.*

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Define $n_j = \binom{n-j}{2}$, $F_{n2} = n_1 + 2$, $F_{ni} = n_2 + i + 2$ for $3 \leqslant i \leqslant n-1$. We then have the following theorem.

THEOREM 1. *Let G be a simple graph with n vertices (but no endvertices) and m edges. Let i be a positive integer such that $2 \leqslant i \leqslant n-1$. Then $m \geqslant F_{ni}$ implies that G is P_i.*

For $i = n-1$ Theorem 1 becomes Theorem 02 thus generalizing it.

Before embarking on the main result we should like to introduce some definitions and notations. Following [5] we shall write property $P_i(u, v)$ *holds in G* meaning that there is an i-path between u and v in G. A t-path p from x_0 to x_t in G will be written as $p = (x_0, x_1, x_2, \ldots, x_t)$. An edge (x_0, x_1) is a special case of a t-path with $t = 1$. The set of edges (vertices) of G will be referred to as E, (V). For a subgraph G' the set of edges (vertices) will be written as $E(G')$, $(V(G'))$. A one-element $\{x\}$ will simply be written as x. For $x \in V$ define $\Gamma(x) = \{y: (x, y) \in E\}$. The expression $\Gamma_{G'}(x)$ will refer to the induced subgraph G'. We have $d(x) = |\Gamma(x)|$, $d_{G'}(x) = |\Gamma_{G'}(x)|$. A graph of order n and size m will be referred to as an *(n, m)-graph*. An *n-graph* is a graph of order n. $m(G) = |E|$, the *size* of G.

Let G_1 and G_2 be symmetric graphs. Then $G_1 \cdot G_2$ denotes the union of G_1 and G_2 such that they have exactly one vertex (and no edge) in common, and $G_1 \mid G_2$ denotes their union such that they have exactly one edge and its endvertices in common.

Let G be an n-graph with an endvertex. Then clearly if $n > 2$, G is not P_i for any i. We therefore consider only connected graphs without endvertices which we shall call henceforth *admissible*. The sizes F_{n2}, $F_{n,n-2}$ and $F_{n,n-1}$ imply admissibility; for all other F_{ni} we have to assume it. However in order to simplify the formulation of our theorems, we require admissibility throughout.

2. The Main Result

We shall first prove Theorem 1 for $i = 2$ and $i = 3$ separately. Then, by using induction on i, we shall prove the general result. For $i = 2$ and $i = 3$ we show the result to be best possible for every n. For $i = n-1$ this is Ore's result and hence is known to be best possible. For $n \leqslant 6$ Theorem 1 is best possible for every i.

I. P_2

We now state

PROPOSITION 1. *An n-graph of size at least F_{n2} is P_2.*

Proof. Let G be an (n, F_{n2})-graph. It is easily seen to be admissible.

It is equally obvious that $n \geqslant 3$. If there is a vertex x_0 of G of degree 2, then $G\backslash x_0$ is a complete $(n-1)$-graph and hence G is easily seen to be P_2. We therefore assume $d(x) > 2$ for all $x \in V$. For $n = 3$ the theorem is clear. Let the theorem hold for $n - 1$. We shall prove it for n. Suppose $P_2(a, b)$ does not hold in G for some a, b of V. Choose a vertex $x_1 \neq b$ which is adjacent to a. Then $(x_1, b) \notin G$ and hence $d(x_1) \leqslant n - 2$. Clearly $G\backslash x_1$ is admissible and $m(G\backslash x_1) \geqslant F_{n2} - (n-2) = F_{n-1,2}$. By the induction hypothesis $G\backslash x_1$ is P_2 and hence $P_2(a, b)$ holds in $G\backslash x_1$, and hence in G, a contradiction. This proves Proposition 1.

We now show our bound in Proposition 1 to be exact. Let G consist of K_{n-2} and two additional vertices a, b such that $\Gamma(a) \cup \Gamma(b) = V$, $\Gamma(a) \cap \Gamma(b) = \varnothing$. The size of such a graph G is $F_{n2} - 1$ and $P_2(a, b)$ does not hold.

II. P_3

We now state

PROPOSITION 2. *An admissible n-graph of size at least F_{n3} is P_3.*

Proof. For $n = 4$ this is clear. Assume the proposition to hold for $n - 1$. We shall prove it for n. Let G be an (n, F_{n3})-graph and let a, b be two of its vertices for which $P_3(a, b)$ does not hold. Let $d(a) \geqslant d(b)$. There is at most one vertex of degree 2 in G. Then $d(a) > 2$. We have

Case 1. There is a vertex x_0 such that $d(x_0) = 2$. Then $G\backslash x_0$ is admissible. We have

Subcase 1.1. $x_0 \neq b$. Then

$$m(G\backslash x_0) = F_{n3} - 2 \geqslant F_{n3} - (n-3) = F_{n-1,3},$$

the inequality stemming from the fact that $n \geqslant 5$. Applying the induction hypothesis we obtain a contradiction.

Subcase 1.2. $x_0 = b$. Since $d(a) \geqslant 3$, there is a vertex x_1 of $\Gamma(a)$ such that $x_1 \neq b$ and $(x_1, b) \notin G$. Also there is a vertex x_2 adjacent to b and distinct from a. By assumption we have $(x_1, x_2) \notin G$. Then b, x_1, x_2 are three distinct vertices not belonging to $\Gamma(x_1)$, so that $|\Gamma(x_1)| \leqslant n - 3$. It follows that $G\backslash x_1$ is admissible since $(x_1, b) \notin G$. We may again apply the induction hypothesis and come to a contradiction.

Case 2. $d(x) > 2$ for all $x \in V$.

Subcase 2.1. There is $x_0 \notin \{a, b\}$ such that $d(x) \leqslant n - 3$. Then by considering $G\backslash x_0$ we again come to a contradiction.

Subcase 2.2. $d(x) \geqslant n - 2$ for all $x \notin \{a, b\}$. Then every vertex of $V\backslash\{a, b\}$

is adjacent either to a or to b. Since $d(b) \geqslant 3$, we have $|\Gamma(b)\backslash a| \geqslant 2$. Let y_1, y_2 be two distinct vertices of $\Gamma(b)\backslash a$. We have

Subcase 2.2.1. Either (a, y_1) or (a, y_2) is not in G. Then there is $x_1 \notin \{b, y_1, y_2\}$ such that $(a, x_1) \in G$. By assumption (x_1, y_1) and (x_1, y_2) are not in G, a contradiction since $d(x_1) \geqslant n - 2$.

Subcase 2.2.2. $\{y_1, y_2\} \subset \Gamma(a)$. Since $n \geqslant 5$, there is a vertex $x_2 \notin \{a, b, y_1, y_2\}$. x_2 is adjacent either to a or to b. It follows that in either case (x_2, y_1) and (x_2, y_2) are not in G, a contradiction again.

This completes the proof of Proposition 2.

We now show Proposition 2 to be best possible by exhibiting for every $n \geqslant 4$ on, an admissible $(n, F_{n3} - 1)$-graph which is not P_3. For $n = 4$ the only (4, 5)-graph is clearly not P_3. Let k be such that for $n = k$ we have shown a $(k, F_{k3} - 1)$-graph G_k which is not P_3. Let us assume that $P_3(a, b)$ does not hold in G_k. We construct G_{k+1} by adding a vertex z to G_k and letting $(z, x) \in E(G_{k+1})$ if $x \neq a, b$. Then G_{k+1} has $k + 1$ vertices and $k_2 + 4 + k - 2 = k_1 + 4$ edges. There is no 3-path from a to b within G_k and there is obviously none through z.

We now pass on to the general case.

III. P_i

THEOREM 1′. *An admissible (n, F_{ni})-graph for $3 \leqslant i \leqslant n - 1$ is P_i.*

Proof. For $i = 3$ this has already been established. For $n < 6$ this is easily checked, the only relevant case being $n = 5$, $i = 4$. We therefore assume $n \geqslant 6$. Let the theorem hold for i', $3 \leqslant i' < i$. We prove it for i. Clearly $i \geqslant 4$. Let G be an (n, F_{ni})-graph and let a, b be two of its vertices for which $P_i(a, b)$ does not hold. Again we assume $d(a) \geqslant d(b)$. Let $d(x_0) = 2$ for some $x_0 \in V$; then $d(x) > 3$ in G for all $x \in V\backslash x_0$. This follows easily from the size of G. It also follows that $i < n - 1$. We now have

Case 1. $(b, x_0) \in E$. Then $d(b) > 2$ (even > 3) and there is an $x_1 \neq a, x_0$, such that $(x_1, b) \in E$. Subgraph $G\backslash\{b, x_0\}$ is an admissible $(n - 2)$-graph of size at least $F_{ni} - n \geqslant F_{n-2,i-1}$ for $n \geqslant 6$. By the induction hypothesis there is an $(i - 1)$-path in $G\backslash\{b, x_0\}$ from a to x_1. Adding (x_1, b) we get an i-path in G from a to b, a contradiction. Put $G\backslash b = G'$, $G'\backslash a = G''$. We now have

Case 2. $(b, x_0) \notin E$. (This includes the possibility that $b = x_0$). Then $d(b) \leqslant n - 2$ and G' is an admissible $(n - 1)$-graph of size at least $F_{ni} - (n - 2) = F_{n-1,i-1}$. Let x_1 be a vertex distinct from a, adjacent to b. By the induction hypothesis there is an $(i - 1)$-path in G' from a to x_1.

Adding (x_1, b) we get an i-path from a to b in G, a contradiction. This proves case 2.

Now let $d(x) > 2$ for all vertices of G. Then G' is admissible. If $d(b) \leqslant n-2$, we proceed as in case 2 above. We therefore assume $d(b) = n - 1$. Then $d(a) = n - 1$. Consider G'. If there is an $(i - 1)$-path in G' from a to some vertex of G', then there is an i-path from a to b in G. We therefore assume that there is no $(i - 1)$-path in G' from a to any other vertex of G'. We shall now distinguish two cases.

Case 1. $i < n - 1$. It is clear that G' is not P_{i-1} and that a is adjacent to all vertices in G'. It follows that G' is P_2. For $i = 4$ this means that G' is P_{i-2}. If $i > 4$, then by the induction hypothesis G' is P_{i-2}, since $F_{ni} - (n - 1) = F_{n-1,i-2}$. Let now y be an arbitrary vertex of G''. Let $p = (a, x_1, x_2, \ldots, x_{i-3}, y)$ be an $(i - 2)$-path from a to y in G'. Suppose $d_{G'}(y) > i - 2$. Then there exists a vertex $x_0 \in G''$ not belonging to p such that $(y, x_0) \in E$. Adding (y, x_0) to p yields an $(i - 1)$-path from a in G', a contradiction. Therefore $d_{G'}(y) \leqslant i - 2$ and hence $d(y) \leqslant i - 1$ for all $y \in G''$. Then we have $2m \leqslant (n - 2)(i - 1) + 2(n - 1)$. This implies $2F_{ni} \leqslant (n - 2)(i - 1) + 2(n - 1)$. Solving the last inequality for i we get $i > n - 2$, a contradiction. This settles case 1.

Case 2. $i = n - 1$. For small n this is clear. We therefore assume case 2 settles for $n - 1$ and prove it for n. As in case 1 we have $d(y) \leqslant i - 1 = n - 2$ for all $y \in G''$. Suppose $d(y) \leqslant n - 3$ for all $y \in G''$. Then $2m \leqslant (n - 2)(n - 3) + 2(n - 1) < (n - 2)(n - 3) + 2(n + 1) = 2F_{n,n-1}$, a contradiction. Then there exists $y_0 \in G$ such that $d(y_0) = n - 2$. Let y_1 be the one vertex of G not adjacent to y_0. By assumption there is no $(i - 2)$-path in G'', for any pair of vertices of G''. Put $G' \cup (y_0, y_1) = \bar{G}'$, $G'' \cup (y_0, y_1) = \bar{G}''$. Using the induction hypothesis with respect to i and n we deduce that $\bar{G}'$ is P_{i-1}. It follows that there is a vertex y_2 in $\bar{G}''$ such that $P_{i-2}(y_2, y_0)$ holds. An $(i - 2)$-path from y_2 to y_0 in $\bar{G}''$ is necessarily of the form $(y_2, x_{\nu_1}, x_{\nu_2}, \ldots, y_1, y_0)$. Our assumption on the degree of y_0 implies that $(y_0, y_2) \in E$ and hence $(y_0, y_2, x_{\nu_1}, x_{\nu_2}, \ldots, y_1)$ is an $(i - 2)$-path in G'', a contradiction. This settles case 2, thus proving Theorem 1′.

Combining Propositions 1 and 2 with Theorem 1′ we arrive at Theorem 1. We now have

COROLLARY 1. (Theorem 02) *If G is a simple (n, m)-graph such that $m \geqslant n_1 + 3$, then G is H-connected.* (*see also* [1, p. 220]).

This follows immediately from Theorem 1′ since the conditions of the corollary imply that G be admissible.

COROLLARY 2. *Let the conditions of Corollary* 1 *for G be satisfied. Then G is P_i for every i, $1 < i < n$.*

Corollary 2 is clearly a generalization of Ore's result.

Remark. We could shorten the proof of Theorem 1′ by using Ore's result in order to dispense with the last case in our proof. However we preferred to prove the last case independently and thus obtain Ore's result as a special case.

We shall now use Theorem 1 in order to prove Theorem 01. (A proof is also given in [5]). We first prove a simple lemma.

LEMMA 1. *An (n, F_{n2})-graph has at most one vertex x such that $d(x) \leqslant \frac{1}{2}(n-1)$.*

Proof. Let G be an (n, F_{n2})-graph. Suppose there are two vertices x, y of degree at most $\frac{1}{2}(n-1)$. Then $m(G\backslash\{x, y\}) \geqslant F_{n2} - n + 1 = n_2 + 1 > n_2$, a contradiction. This proves the lemma.

We now turn to the proof of the theorem. For $n \leqslant 3$ the theorem is clear. We therefore assume $n \geqslant 4$. Let G be an (n, F_{n2})-graph. Let x be a vertex of G of minimal degree. Let y be any vertex adjacent to x. Since $F_{n2} = F_{n,n-2}$ for $n \geqslant 4$ and the size F_{n2} implies admissibility of G, G is P_{n-2} and hence there is an $(n-2)$-path from x to y in G. Adding (x, y) to p yields an $(n-1)$-cycle C in G. Let z be the vertex of G left out in C. Using the lemma and the minimal property of x we deduce that $d(z) > \frac{1}{2}(n-1)$. Then $\Gamma(z)$ contains a pair of neighbouring vertices of C and so z completes C to a hamiltonian cycle in G, thus proving Theorem 01.

It was already mentioned that up to $n = 6$ our bound is exact for every i. For example the (6, 11)-graph No. 2 in [7, Appendix I] is P_2 but not P_4 whereas the (6, 11)-graph No. 7 is P_4 but not P_2.

On the other hand it turns out that already for $n = 7$, Theorem 1 is not best possible for $i = 4, 5$. We therefore proceed to establish the best bounds for every $n \geqslant 7$ and every i, $4 \leqslant i \leqslant n - 2$. We start with a lemma.

LEMMA 2. *Let G be an admissible $(n, n_2 + 4)$-graph. Let $(x, y) \in E$ be such that $d(x) + d(y) \leqslant n - 1$. Then G is P_i for $4 \leqslant i \leqslant n - 3$.*

Proof. Let G be a graph satisfying the conditions of Lemma 2 and let a, b be two arbitrary vertices of G. We show $P_i(a, b)$. Put $n_2 + i = f(n, i)$. We have

Case 1. $b \in \{x, y\}$. Consider $G' = G\backslash\{x, y\}$. This is an $(n-2, f(n, 4) - n + 2)$-graph. But $f(n, 4) - n + 2 = F_{n-2,n-3}$ so that by Corollary 2, G' is P_i for $2 \leqslant i \leqslant n - 3$. If $a \in \{x, y\}$, choose two distinct vertices u, v of G' such that $\{(u, a), (v, b)\} \subset E$. Since $P_i(u, v)$ holds in G' for $2 \leqslant i \leqslant n - 3$, $P_i(a, b)$ holds in G for $4 \leqslant i \leqslant n - 1$. If $a \notin \{x, y\}$ put $b = y$. Choose a vertex $z \notin \{a, b, x\}$ such that $(z, b) \in E$, otherwise $(z, x) \in E$. Again G' is P_i

for $2 \leqslant i \leqslant n-3$, so that $P_i(a, z)$ holds. Since either $(z, b) \in E$ or (z, x, b) is a 2-path in G, we have $P_i(a, b)$ in G for $4 \leqslant i \leqslant n-2$.

Case 2. $b \notin \{x, y\}$. Then $P_i(a, b)$ holds in G' for $2 \leqslant i \leqslant n-3$, so that $P_i(a, b)$ holds in G. This proves the lemma.

We now show that an admissible (9, 25)-graph is P_5 and P_6. Let G be an admissible (9, 25)-graph and let a, b be two arbitrary vertices of G. We show that $P_5(a, b)$ holds. Let $d(a) \geqslant d(b)$. We have

Case 0. There is $x \in V\backslash\{a, b\}$ such that $d(x) \leqslant 3$. If $d(x) = 3$ and there is a vertex y adjacent to x with $d(y) = 2$, then $G\backslash\{x, y\} = K_7$, so that it is easily checked that G is P_5. We therefore assume $G\backslash x$ to be admissible. It is either a (8, 22)- or a (8, 23)-graph. Then by Theorem 1, $G\backslash x$ is P_5, so that $P_5(a, b)$ holds in $G\backslash x$ and hence in G.

We now assume that for all $x \in V\backslash\{a, b\}$ we have $d(x) > 3$. Then $G\backslash b$ is admissible.

Case 1. $d(b) \leqslant 6$. Then $G\backslash b$ is admissible with $m(G\backslash b) \geqslant 19$ and hence $G\backslash b$ is P_4, so that $P_5(a, b)$ holds in G.

Case 2. $d(b) = 7$. Put $\Gamma(b)\backslash a = X$. Let x_0 be a vertex of X such that $d(x_0)$ is minimal in X. Considering case 0 we may assume that $G\backslash\{b, x_0\}$ is admissible. If $d(x_0) \leqslant 5$, then $m(G\backslash\{x_0, b\}) \geqslant 25 - 11 = 14$, so that $G\backslash\{x_0, b\}$ is P_4 and hence $P_5(a, b)$ holds in G.

Case 3. $d(b) = 8$. Then $d(a) = 8$ and $m(G\backslash\{a, b\}) = 25 - 15 = 10$, so that $G\backslash\{a, b\}$ is a (7, 10)-graph which is known to have a 3-path (x, y, z, u). Then (a, x, y, z, u, b) is a 5-path in G.

We now assume $d(x_0) \geqslant 6$. Since $m(G) = 25$ we have $P_4(a, b)$. Let (a, x_1, x_2, x_3, b) be a 4-path in G. Let the other vertices of G be $Y = \{y_1, y_2, y_3, y_4\}$. We have $d(x_3) \geqslant 6$ and hence x_3 is adjacent to at least 2 vertices of Y. But b is adjacent to at least 3 vertices of Y. The pigeonhole principle then implies that there is $y_j \in Y$, such that $\{(y_j, x_3), (y_j, b)\} \subset E$. Then $(a, x_1, x_2, x_3, y_j, b)$ is a 5-path in G. Property P_5 is thus proved.

We now prove that $P_6(a, b)$ holds in the (9, 25)-graph G. By what we have just shown, $P_5(a, b)$ holds in G. Let $(a, x_1, x_2, x_3, x_4, b)$ be a 5-path in G. Let the other vertices of G be $Y = \{y_1, y_2, y_3\}$.

If for some vertex $v \in G$, $d(v) = 2$, then $G\backslash v$ is an (8, 23)-graph and hence admissible. If $v = a$ or $v = b$, $G\backslash v$ is P_5 by Theorem 1. If $v \notin \{a, b\}$, then $G\backslash v$ is P_6 by Theorem 1. In both cases we obtain $P_6(a, b)$. We therefore assume $d(v) \geqslant 3$ for $v \in V$ and hence $G\backslash v$ is admissible. If for some $j_0 = 1, 2, 3$, $d(y_{j_0}) \leqslant 4$, then $G\backslash y_{j_0}$ is an admissible 8-graph of size at least 21 which is easily shown to be P_6, so that $P_6(a, b)$ holds in $G\backslash y_{j_0}$ and hence in G. We now assume $d(y_j) \geqslant 5$ for $j = 1, 2, 3$. Put $U = V\backslash Y$. We have clearly $|\Gamma(y_j) \cap U| \geqslant 3$. If for some j we have $|\Gamma(y_j) \cap U| > 3$, then it is easily seen that y_j is adjacent to two neighbouring vertices of the 5-path, so that $P_6(a, b)$ holds. We thus have only to consider the case $|\Gamma(y_j)\, U| = 3$. Then $d(y_j) \geqslant 5$ implies $\{(y_1, y_2), (y_2, y_3), (y_3, y_1)\} \subset E$. It can easily be checked that any set of $\Gamma(y_j)$ gives a 6-path in G from a to b. This proves P_6 for G. In a similar manner we can show that an admissible (8, 20)-graph is P_5. On the other hand $K_5 \mid K_5$ is an admissible (8, 19)-graph which is not P_5.

We now prove

THEOREM 2′. *An admissible $(n, n_2 + 4)$-graph, $n \geqslant 9$, is P_i for $4 \leqslant i \leqslant n-3$.*

For $i = 4$ this has already been shown. Also for $n = 9$, $i = 4, 5, 6$. We therefore assume the statement true for $n' < n$ and every i, $4 \leqslant i \leqslant n' - 3$ and also for n and every i', $4 \leqslant i' < i$, and prove it for n and i.

Let G be an admissible $(n, n_2 + 4)$-graph and let a, b be two arbitrary vertices of G. We show $P_i(a, b)$. We assume $i > 4$, $n > 9$, $d(a) \geqslant d(b)$. By assumption $P_{i-1}(a, b)$ holds. Let $(a, x_1, x_2, \ldots, x_{i-2}, b)$ be an $(i - 1)$-path in G from a to b. Let Y be the set of vertices of G not belonging to the $(i - 1)$-path. First let $4 < i < n - 3$. Let $y \in Y$. We have

Case 1. $d(y) \leqslant n - 3$. By Lemma 2 we may assume $G' = G \backslash Y$ admissible. We know that G' is an $(n - 1)$-graph of size at least $(n - 1)_2 + 4$ and hence is P_i (for $i < (n - 1) - 2 = n - 3$). It follows that $P_i(a, b)$ holds in G' and hence in G.

Case 2. $d(y) \geqslant n - 2$. Then there are two consecutive vertices of the $(i - 1)$-path adjacent to y, thus forming an i-path in G from a to b.

Now let $i = n - 3$. We have

Case 1. $d(y) \leqslant n - 5$ for some vertex $y \in Y$. Then, similarly to the previous case 1, $G' = G \backslash y$ is an $(n - 1)$-graph of size at least $(n - 1)_2 + 6$ so that $P_i(a, b)$ holds in G' and hence in G.

Case 2. $d(y) \geqslant n - 4$ for every $y \in Y$. For $n = 10$ the treatment is similar to that of $n = 9$. For $n > 10$ the argument is as in case 2 for $i < n - 3$. This completes the proof.

We now have

PROPOSITION 3. *An admissible $(n, n_2 + 6)$-graph is P_{n-2}.*

Proof. For $n \leqslant 7$ this may be checked ([4], [7]). Let $n > 7$. Let the theorem be true for graphs of order less than n. Let G be an admissible $(n, n_2 + 6)$-graph and let a, b be two arbitrary vertices of G. We show $P_{n-2}(a, b)$ to hold. Let $d(a) \geqslant d(b)$. We have

Case 1. $d(b) \leqslant n - 3$. Then $m(G \backslash b) \geqslant n_2 + 6 - (n - 3) = \frac{1}{2}(n^2 - 7n + 24) = (n - 1)_2 + 6$ and hence by the induction hypothesis $G \backslash b$ is $P_{(n-1)-2} = P_{n-3}$.

Since G is admissible, b is adjacent to some vertex $u \neq a$ in G, which ensures an $(n-2)$-path from a to b.

Case 2. $d(b) \geqslant n-2$. We have already mentioned that an $(8, 20)$-graph is P_5. For $n \geqslant 9$ it follows from Theorem 2′ that G is P_{n-3}, so that there is a path $p_{n-3}(a, b)$. Let $x, y \in V\backslash\{a, b\}$ such that $x, y \notin p_{n-3}(a, b)$ with $d(x) \geqslant d(y)$. Suppose $d(y) \leqslant 3$. Then $m(G\backslash y) \geqslant n_2 + 3 = (n-1)_1 + 3$, so that $G\backslash y$ is P_{n-2} by Theorem 1′. Then $P_{n-2}(a, b)$ in $G\backslash y$ and hence in G. We therefore assume $d(y) > 3$, and hence $d(x) > 3$.

If $d(x) > \frac{1}{2}(n+1)$, then x is adjacent to two neighbouring vertices of $p_{n-3}(a, b)$, implying $P_{n-2}(a, b)$ so that we may assume $d(x) \leqslant \frac{1}{2}(n+1)$. Then $d(y) \leqslant \frac{1}{2}(n+1)$. The degree of b implies $(b, x) \in E$ or $(b, y) \in E$. Without loss of generality we may assume $(b, x) \in E$.

Subcase 2.1. $d(b) = n-2$.

$$\begin{aligned} m(G\backslash\{b, x\}) &\geqslant n_2 + 6 - (n-2) - \tfrac{1}{2}(n+1) \\ &= \tfrac{1}{2}(n^2 - 8n + 25) \geqslant (n-2)_2 + 6 \qquad \text{for} \quad n \geqslant 7, \end{aligned}$$

and hence by the induction hypothesis $G\backslash\{b, x\}$ is P_{n-4}. Since $d(x) > 3$, there is a vertex z in $G\backslash\{a, b\}$ such that $(x, z) \in E$. Let $p(a,..., z)$ be an $(n-4)$-path in $G\backslash\{x, b\}$. Add to it (z, x, b) thus forming an $(n-2)$-path in G from a to b.

Subcase 2.2. $d(b) = n-1$. Then $d(a) = n-1$.

Suppose that $d(x) > \frac{1}{2}(n-1)$. If $(x, y) \notin E$, then x is adjacent to two neighbouring vertices of $p_{n-3}(a, b)$, thus forming an $(n-2)$-path. We therefore assume $(x, y) \in E$. We also assume that x is not adjacent to two neighbouring vertices of $p_{n-3}(a, b)$. Put $p_{n-3}(a, b) = (a, u_1, u_2, ..., v_2, v_1, b)$. Clearly $(x, u_1) \notin E$. If $(x, u_2) \in E$, we have $(a, y, x, u_2, u_3, ..., v_2, v_1, b)$ an $(n-2)$-path in G. If $(x, u_2) \notin E$, then necessarily $(x, v_2) \in E$ and $(a, u_1, u_2, ..., v_3, v_2, x, y, b)$ is an $(n-2)$-path in G. We therefore assume $d(x) \leqslant \frac{1}{2}(n-1)$. As in subcase 2.1 we have

$$\begin{aligned} m(G\backslash\{b, x\}) &\geqslant n_2 + 6 - (n-1) - \tfrac{1}{2}(n-1) \\ &= \tfrac{1}{2}(n^2 - 8n + 25) \geqslant (n-2)_2 + 6 \qquad \text{for} \quad n \geqslant 7 \end{aligned}$$

and the rest follows as in subcase 2.1.

Proposition 3 is thus proved. Considering $K_{n-2} \mid K_4$ it becomes clear that Proposition 3 is best possible.

PROPOSITION 4. *An admissible $(n, n_2 + 4)$-graph, $n \geqslant 7$, is P_4.*

Proof. For $n = 7$ this may be checked [4]. Let Proposition 4 hold for all graphs of order greater than 6 and less than n. Let G be an admissible

$(n, n_2 + 4)$-graph, $n > 7$, and let a, b be two arbitrary vertices of G. We show that $P_4(a, b)$ holds in G.

Suppose G has two vertices x, y of degree two. This is only possible if $G\backslash\{x, y\} = K_{n-2}$ and $(x, y) \notin E$. Such a G is clearly P_4 (since $n > 6$). We now assume that G has at most one vertex of degree two. Let x_0 be a vertex of minimal degree in G among $V\backslash\{a, b\}$. Let us again assume $d(a) \geqslant d(b)$. We have

Case 1. $d(b) = 2$. Let x_1 be the vertex adjacent to b distinct from a. Consider $G\backslash b$. This is an $((n-1), (n-1)_1 + 2)$-graph and hence is P_i for all i, $1 < i \leqslant n - 2$ by Corollary 2. In particular $G\backslash b$ is P_3. Since $P_3(a, x_1)$ holds in $G\backslash b$, $P_4(a, b)$ holds in G.

Case 2. $d(x_0) \leqslant n - 3$ and $d(b) > 2$. Then $G\backslash x_0$ is an admissible $(n-1)$-graph of size at least $(n-1)_2 + 4$ and hence is P_4, so that $P_4(a, b)$ holds in $G\backslash x_0$ and hence in G.

Case 3. $d(x_0) \geqslant n - 2$ and $d(b) > 2$. Then every vertex x of $V\backslash\{a, b\}$ is non-adjacent in G to at most one vertex of $V\backslash x$. Choose $x_1 \in \Gamma(a)$ and $x_2 \in \Gamma(b)$ such that $|\{x_1, x_2, a, b\}| = 4$. This is possible since $d(a) \geqslant d(b) > 2$. Let x_3, x_4, x_5 be three other vertices of G, but otherwise arbitrary. Since x_1 and x_2 are non-adjacent to at most one vertex each, at least one of the three is adjacent to both x_1 and x_2. Let this vertex be x_3. Then (a, x_1, x_3, x_2, b) is a 4-path in G. Proposition 4 is thus proved.

Consider $G = K_{n-2} \cdot K_3$. This is an n-graph of size $n_2 + 3$ and is clearly not P_4 so that Proposition 4 is best possible.

We may now recapitulate our results by stating

THEOREM 2. *Let G be an admissible (n, f_{ni})-graph, $n \geqslant 9$ and*

$$f_{ni} = \begin{cases} n_1 + 2 & \text{for } i = 2 \\ n_2 + 5 & \text{for } i = 3 \\ n_2 + 4 & \text{for } 4 \leqslant i \leqslant n - 3 \\ n_2 + 6 & \text{for } i = n - 2 \\ n_1 + 3 & \text{for } i = n - 1. \end{cases}$$

Then G is P_i.

For $i = 2, 3, 4, n-2, n-1$ we know Theorem 2 to be the best possible. The critical graph $K_3 \cdot K_{n-2}$ from Proposition 4 serves also for $4 < i \leqslant n - 3$. It follows that Theorem 2 is exact for every $n \geqslant 9$ and i, $2 \leqslant i \leqslant n - 1$.

3. Large Cycles and Pancyclicity

A theorem of Ore states that an $(n, n_1 + 2)$-graph is hamiltonian [8]. Considering $K_2 + \overline{K}_3$ we find that an $(n, n_1 + 1)$-graph even though admissible may not be hamiltonian. However this turns out to be the only admissible exception. We have the following

THEOREM 3. *An admissible* $(n, n_1 + 1)$-*graph* (except for $K_2 + \bar{K}_3$) *is hamiltonian.*

Proof. Let G be an admissible $(n, n_1 + 1)$-graph. Up to $n = 6$ the theorem may easily be verified. We therefore assume $n \geqslant 7$. Then $n_1 + 1 \geqslant n_2 + 6$, and hence by Proposition 3, G is P_{n-2}. It therefore contains an $(n-1)$-cycle. Let the additional vertex of G be u. Let v be a vertex adjacent to u and let x be a neighbouring vertex to v in the cycle. If $(u, x) \in E$, then G is hamiltonian. We therefore assume $(u, x) \notin E$. We now have

Case 1. $d(u) + d(x) < n - 1$. Then $G \backslash \{u, x\}$ is of size at least $n_1 + 1 - (n-2) > n_2$, a contradiction.

Case 2. $d(u) + d(x) > n - 1$. Clearly G has a hamiltonian path, denote it by $h = (u, v = x_1, x_2, \ldots, x_{n-1} = x)$. Let $\Gamma(u) = \{x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_s}\}$ and $\Gamma(x) = \{x_{\mu_1}, x_{\mu_2}, \ldots, x_{\mu_t}\}$ be subsequences of h. We have $\nu_s + \mu_t > n - 1$ and hence there are i, j such that $\mu_i + 1 = \nu_j$, so that $(v, x_1, x_2, \ldots, x_{\mu_i}, x, x_{n-2}, \ldots, x_{\nu_j}, v)$ is a hamiltonian cycle. (This is practically a reproduction of one of the stages in the proof of Posa's Theorem [10]. See also [7, p. 66]).

Case 3. $d(u) + d(x) = n - 1$. Then $G \backslash \{u, x\} = K_{n-2}$. It follows by the pidgeon-hole principle that there is a vertex w of K_{n-2} which is adjacent both to u and x. We have $n - 1 \geqslant 6$, so that there are distinct vertices y, z of $G \backslash \{u, x, w\}$ such that $\{(z, u), (x, y)\} \subset E$. Then $(z, u, w, x, y, \ldots, z)$ is a hamiltonian cycle in G. The last part of the sequence is conceivable since we are dealing with a complete subgraph. This proves Theorem 3.

We have

COROLLARY 3 (Theorem 01 [8]). *An* (n, m)-*graph with* $m \geqslant n_1 + 2$ *is hamiltonian.*

The proof is immediate since $n_1 + 2$ implies admissibility. This last result of Ore was also proved in [5].

An n-graph is *pancyclic* if it contains cycles of every length c, $3 \leqslant c \leqslant n$. By a result of Bondy [2] a hamiltonian $(n, n_2 + 4)$-graph is pancyclic. This also follows easily from a result of Woodall [11, Corollary 11.1]. A special case is obtained in [2].

Replacing *admissibility* by *hamiltonicity*, which implies admissibility, we may deduce that in a hamiltonian $(n, n_2 + 4)$-graph every edge belongs to a cycle of length c, for $5 \leqslant c \leqslant n - 2$. Adding two edges to the n-graph the result extends to $3 \leqslant c \leqslant n - 1$.

Using Ore's Theorem 01 Bondy immediately derived than an $(n, n_1 + 2)$-

TABLE I

$n \backslash i$	1	2	3	4	5	6	7	8	
3	3	3							
4	6	5	6						
5	10	8	8	9					
6	15	12	11	12	13				
7	21	17	15	14	16	18			
8	28	23	20	19	20	21	24		
9	36	30	26	25	25	25	27	31	
⋮	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	
n	n_0,	$n_1 + 2$,	$n_2 + 5$,	$n_2 + 4$,	. . . . .	. . . . .,	$n_2 + 4$,	$n_2 + 6$,	$n_1 + 3$.

(By definition $n_j = \binom{n-j}{2}$).

graph is pancyclic [3]. By adding one edge, much more can be said. We have

COROLLARY 4. *Let G be an $(n, n_1 + 3)$-graph. Then, given any edge e of G and any integer k, $3 \leqslant k \leqslant n$, there is in G a cycle of length k containing e.*

We conclude with supplying a table showing $m(n, i)$, the least upper bound for the size of a graph ensuring P_i for given n and i, $n > 2$, $1 \leqslant i \leqslant n - 1$. The graphs in question are always admissible for $i = 1, 2, n - 1$; otherwise they have to be assumed admissible. The case $i = 1$ is included for the sake of completeness.

REFERENCES

1. C. BERGE, "Graphs and Hypergraphs," North-Holland, Amsterdam, 1973.
2. J. A. BONDY, Large cycles in graphs, *Discrete Math.* **1** (1971), 121–132.
3. J. A. BONDY, Pancyclic graphs, *J. Combinatorial Theory B* **11** (1971), 80–84.
4. D. W. CROWE, Graph diagrams of the 7-graphs, 1953, private communication.
5. R. J. FAUDREE, C. C. ROUSSEAU, AND R. H. SCHELP, Theory of path length distributions. I, *Discrete Math.* **6** (1973), 35–52.
6. R. J. FAUDREE AND R. H. SCHELP, Path connected graphs, *Acta Math. Acad. Sci. Hungar.* **25** (1974), 313–319.
7. F. HARARY, "Graph Theory," Addison–Wesley, Reading, Mass., 1972.
8. O. ORE, Arc coverings of graphs, *Ann. Mat. Pura Appl.* **55** (1961), 315–321.

9. O. Ore, Hamilton-connected Graphs, *J. Math. Pures Appl.* **42** (1963), 21–27.
10. L. Posa, A theorem concerning Hamilton lines, *Magyar Tud. Akad. Mat. Kutato Int. Közl.* **7** (1962), 225–226.
11. D. R. Woodall, Sufficient conditions for circuits in graphs, *Proc. London Math. Soc.* **24** (1972), 739–755.